Semester 2026A, Exam A, Q2

Problem. Let $D\colon M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ be a volume function, and let $A \in M_{n\times n}(\mathbb{F})$.
We will denote by ε the elementary row operation $R_i \to R_i + cR_j$, where $1 \le i, j \le n$, $i \ne j$, and $c \in \mathbb{F}$.
Prove that $D(\varepsilon(A)) = D(A)$.

Proof. Let us mark A with $\begin{pmatrix} R_1 \\ \vdots \\ R_j \\ \vdots \\ R_i \\ \vdots \\ R_n \end{pmatrix}$ (we assume without loss of generality that $j < i$).
Then the matrix A after operation ε will be

$$\begin{pmatrix} R_1 \\ \vdots \\ R_j \\ \vdots \\ R_i + cR_j \\ \vdots \\ R_n \end{pmatrix}$$

Since a volume function is in particular a multi-linear function,

$$D\left(\begin{pmatrix} R_1 \\ \vdots \\ R_j \\ \vdots \\ R_i + cR_j \\ \vdots \\ R_n \end{pmatrix}\right) = D\left(\begin{pmatrix} R_1 \\ \vdots \\ R_j \\ \vdots \\ R_i \\ \vdots \\ R_n \end{pmatrix}\right) + D\left(\begin{pmatrix} R_1 \\ \vdots \\ R_j \\ \vdots \\ cR_j \\ \vdots \\ R_n \end{pmatrix}\right) = D\left(\begin{pmatrix} R_1 \\ \vdots \\ R_j \\ \vdots \\ R_i \\ \vdots \\ R_n \end{pmatrix}\right) + cD\left(\begin{pmatrix} R_1 \\ \vdots \\ R_j \\ \vdots \\ R_j \\ \vdots \\ R_n \end{pmatrix}\right)$$

Recall that for a matrix B and for the elementary row operation $R_w \leftrightarrow R_k$ which we mark with ε', $D(\varepsilon'(B)) = -D(B)$.

$$\therefore D\left(\begin{pmatrix}R_1\\\vdots\\R_j\\\vdots\\R_i\\\vdots\\R_n\end{pmatrix}\right)+cD\left(\begin{pmatrix}R_1\\\vdots\\R_j\\\vdots\\R_j\\\vdots\\R_n\end{pmatrix}\right)=D\left(\begin{pmatrix}R_1\\\vdots\\R_j\\\vdots\\R_i\\\vdots\\R_n\end{pmatrix}\right)+c\cdot -D\left(\begin{pmatrix}R_1\\\vdots\\R_j\\R_j\\\vdots\\\vdots\\R_n\end{pmatrix}\right)$$

And since D is a volume function, it's an alternating function in particular, which means it returns zero on a matrix that has two adjacent identical rows:

$$c\cdot -D\left(\begin{pmatrix}R_1\\\vdots\\R_j\\R_j\\\vdots\\R_n\end{pmatrix}\right)=c\cdot -0=0$$

Hence

$$D\left(\begin{pmatrix}R_1\\\vdots\\R_j\\\vdots\\R_i+cR_j\\\vdots\\R_n\end{pmatrix}\right)=D\left(\begin{pmatrix}R_1\\\vdots\\R_j\\\vdots\\R_i\\\vdots\\R_n\end{pmatrix}\right)$$

Which means $D(\varepsilon(A))=D(A)$ as required. □

This is an exam problem I translated and solved. The original exam was written by the Linear Algebra 1 course lecturers at HUJI.