Semester 2026A, Exam A, Q6

Problem. Let $\mathbb{F}$ be a field and let $V = \mathbb{F}[X]_{\leq 3}$.
We shall define the linear map $T\colon V \to V$ by

$$(T(p))(X) = p(X) - p(1 - X).$$

Find bases for $\ker(T)$ and for $\mathrm{Im}(T)$. Prove your answers.

Proof. The standard basis of V is $\left(1, x, x^2, x^3\right)$.
Let us compute T on the standard basis.

$$T(1) = 1 - 1 = 0$$

$$T(x) = x - (1 - x) = 2x - 1$$

$$T(x^2) = x^2 - (1 - x)^2 = x^2 - (1 - 2x + x^2) = 2x - 1$$

$$T(x^3) = x^3 - (1 - x)^3 = x^3 - (1 - x)^2(1 - x) = x^3 - (1 - 2x + x^2)(1 - x) =$$

$$x^3 - (1 - 3x + 3x^2 - x^3) = 2x^3 - 3x^2 + 3x - 1$$

Let us recall that if S spans V and the domain of a linear transformation T is V then $T(S)$ spans $\mathrm{im}(T)$.
This means that

$$\mathrm{Span}\left(0, 2x - 1, 2x - 1, 2x^3 - 3x^2 + 3x - 1\right) = \mathrm{Im}(T)$$

Let us recall that for a span of a sequence of vectors, if one vector is a linear combination of the others, we can remove it and not change the span.
In any field,

$$0 \cdot (2x - 1) = 0 \quad \wedge 1 \cdot (2x - 1) = 2x - 1$$

$$\therefore \quad \mathrm{Span}\left(2x - 1, 2x^3 - 3x^2 + 3x - 1\right) = \mathrm{Im}\,(T)$$

Let $a, b \in \mathbb{F}$

$$a(2x - 1) + b(2x^3 - 3x^2 + 3x - 1) = 0$$

$$2bx^3 - 3bx^2 + (3b + 2a)x + (-a - b) \cdot 1 = 0$$

Since $\left(1, x, x^2, x^3\right)$ is a basis of V there is exactly one way to reach any element of V as a linear combination of it.
This gives us

$$2b = 0 \wedge -3b = 0 \quad \therefore \quad b = 3b - 2b = 0$$

$$-a - b \underset{\text{plug in b=0}}{=} -a \underset{-a-b=0}{=} 0 \therefore a = 0$$

Recall the claim that a sequence is linearly independent if the only way to reach zero as a linear combination of its elements is by setting all of the scalars to zero, then $\left(2x - 1, 2x^3 - 3x^2 + 3x - 1\right)$ is linearly independent.
This means $\left(2x - 1, 2x^3 - 3x^2 + 3x - 1\right)$ is a base for $\mathrm{Im}(T)$ since it is linearly independent and spans it.
Now the kernel. It's all of the values that are mapped to 0_V. i.e. since we know the basis for V, every

$$p = \alpha \cdot 1 + \beta \cdot x + \gamma \cdot x^2 + \delta \cdot x^3 \quad s.t. \quad T(p) = 0_V$$

We already have a basis for $\mathrm{Im}\, T$, which means there is one way to represent $T(p)$ as a linear combination of the vectors in the basis:

$$T(p) = T\left(\alpha \cdot 1 + \beta \cdot x + \gamma \cdot x^2 + \delta \cdot x^3\right) = 0$$

This is a linear transformation so we get

$$T(\alpha \cdot 1) + T(\beta x) + T(\gamma x^2) + T(\delta x^3) = 0$$
$$\alpha \cdot T(1) + \beta \cdot T(x) + \gamma \cdot T(x^2) + \delta \cdot T(x^3) = 0$$

$$\alpha \cdot 0 + \beta \cdot (2x - 1) + \gamma \cdot (2x - 1) + \delta \cdot \left(2x^3 - 3x^2 + 3x - 1\right) = 0$$

$$(\beta + \gamma)(2x - 1) + \delta\left(2x^3 - 3x^2 + 3x - 1\right) = 0$$

And since we know that the only way to reach zero as a linear combination of $(2x - 1, 2x^3 - 3x^2 + 3x - 1)$ is by taking all of the scalars as zero, we get

$$\beta = -\gamma, \quad \delta = 0$$

Therefore

$$\ker(T) = \left\{\alpha + (-\gamma) \cdot x + \gamma \cdot x^2 \mid \alpha, \gamma \in \mathbb{F}\right\} = \left\{\alpha \cdot 1 + \gamma \cdot (x^2 - x) \mid \alpha, \gamma \in \mathbb{F}\right\} = \mathrm{Span}\left(1, x^2 - x\right)$$

$$a \cdot 1 + b \cdot (x^2 - x) = 0 \iff bx^2 - bx + a = 0$$

Again, there is a unique way to represent 0 using the base $(1, x, x^2, x^3)$ and this gives us $b = 0, a = 0$ therefore this sequence spans the kernel and is linearly independent, i.e. a base for the kernel. $\square$

> This is an exam problem I translated and solved. The original exam was written by the Linear Algebra 1 course lecturers at HUJI.