Semester 2026A, Exam A, Q7

Problem. Let V be a vector space with $\dim V = n$, and let $T\colon V \to V$ be a linear map such that $\mathrm{Im}(T) \cap \ker(T) \neq \{0_V\}$. Prove that $\mathrm{rk}(T \circ T) \leq n - 2$.

Proof.

$$\mathrm{rk}(T \circ T) = \dim\left(\mathrm{Im}(T \circ T)\right) = \dim\left(\{T\left(T\left(v\right)\right) \mid v \in V\}\right) = \dim\left(\{T(v) \mid v \in \mathrm{Im}T\}\right)$$

Let us look at the function $T' : \mathrm{Im}(T) \to V$ which is defined by $\forall x \in \mathrm{Im}(T), T'(x) = T(x)$ (It's also a linear transformation from inheritance)

$$x \in \ker(T') \iff x \in \mathrm{Im}(T) \wedge T'(x) = 0 \iff x \in \mathrm{Im}(T) \wedge T(x) = 0 \iff x \in \mathrm{Im}(T) \wedge x \in \ker(T)$$

Then

$$\ker(T') = \mathrm{im}(T) \cap \ker(T) \neq \{0_V\}$$

Which means

$$\dim\left(\ker(T')\right) \geq 1$$

Furthermore, we know that $\mathrm{Im}(T) \cap \ker(T) \neq \{0_V\}$ so in particular $\ker(T) \neq \{0_V\}$ and $\dim\left(\ker(T)\right) \geq 1$ as well.
According to the second dimensions theorem,

$$\dim\left(\mathrm{Im}(T)\right) = \dim\left(V\right) - \dim\left(\ker(T)\right) \leq n - 1$$

and

$$\dim\left(\mathrm{Im}(T')\right) = \dim\left(\mathrm{Im}(T)\right) - \dim\left(\ker(T')\right) \leq \dim\left(\mathrm{Im}(T)\right) - 1 \leq n - 1 - 1 = n - 2$$

Let us notice that

$$\mathrm{Im}(T') = \{T'(x) \mid x \in \mathrm{Im}(T)\} = \{T(x) \mid x \in \mathrm{Im}(T)\} = \{T\left(T\left(x\right)\right) \mid x \in V\} = \mathrm{Im}(T \circ T)$$

Therefore

$$\mathrm{rk}(T \circ T) = \dim\left(\mathrm{Im}(T \circ T)\right) = \dim\left(\mathrm{Im}(T')\right) \leq n - 2$$

□

This is an exam problem I translated and solved. The original exam was written by the Linear Algebra 1 course lecturers at HUJI.