

Semester 2026A, Exam A, Q8

Problem. Let $A, B \in M_{m \times n}(\mathbb{R})$ be matrices and let $b \in \mathbb{R}^m$, such that the system of linear equations $Ax = b$ has a unique solution. Which of the following claims are true? If a claim is true, prove it; if not, give a counterexample.

1. If there exists an invertible matrix $P \in M_{m \times m}(\mathbb{R})$ such that $B = PA$, then the system of linear equations $Bx = b$ has a unique solution.
2. If there exists an invertible matrix $P \in M_{n \times n}(\mathbb{R})$ such that $B = AP$, then the system of linear equations $Bx = b$ has a unique solution.

Analyzing the question:

By the guidance of a gifted mathematical monk I happened to come across upon my ways, we shall approach this question from the viewpoint of functions, as an important essence of linear algebra is that every matrix can be represented as a linear transformation.

In particular, the matrix A defines a linear transformation $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ that operates according to the rule $T_A(x) = Ax$.

The given data devilishly appears as one fact but is actually two: $Ax = b$ has a solution, and this solution is unique. Its existence, translated to our language of choice, means that $b \in \text{Im}(T_A)$, because there exists a solution s.t. $T_A(x) = Ax = b$.

For uniqueness, let U denote the solution set of the homogeneous system $Ax = 0$, and recall that $U = \ker(T_A)$ by definition. Let us recall the theorem: if $Ax = b$ has a solution and v is any solution of it, then its solution set is exactly $v + U$.

Our system has a solution, and its solution set is a single vector, say $\{v\}$.

Applying the theorem, $v + U = \{v\}$, and hence $U = \{0\}$.

Let us recall that a linear transformation is injective if and only if its kernel is zero, which means T_A is injective.

We shall also recall that if $G \in M_{k \times k}(\mathbb{R})$ is invertible, then $T_G : \mathbb{R}^k \rightarrow \mathbb{R}^k$ is bijective.

With these arrows in our quiver we shall bravely approach this question.

Part 1:

Part 1 asks us (and quite impolitely so) to study the linear transformation $T_P \circ T_A$, since $B = PA$ gives $T_B = T_P \circ T_A$. Do we know that there exists exactly one $x \in \mathbb{R}^n$ s.t. $T_P(T_A(x)) = b$? To quote Monty Python's argument skit, not necessarily (T_A can choose to satisfy this in his spare time).

Note first what cannot go wrong.

T_A is injective, and T_P is bijective, so $T_B = T_P \circ T_A$ is a composition of injective maps and is therefore injective.

Hence $Bx = b$ has at most one solution, one way or another.

Only existence can fail, and fail it does.

Let us see where the counter-example can live.

If $m = n$, then A is injective and square, hence invertible, so $B = PA$ is invertible as well and $Bx = b$ has a unique solution for every b .

Moreover, because T_A is injective, we must choose $n \leq m$ - lest this attribute will not be satisfied.

Therefore, we must choose $m > n$.

Let us choose $A = b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

First, we verify that the counter-example fulfills the requirements.

Here $A \in M_{2 \times 1}(\mathbb{R})$ and $b \in \mathbb{R}^2$, and $Ax = b$ means

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \therefore x = 1$$

so $x = 1$ is indeed the unique solution.

P is invertible: computing P^{-1} works

$$\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array} \right)$$

so $P^{-1} = P$, which we confirm by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} (0 \cdot 0 + 1 \cdot 1) & (0 \cdot 1 + 1 \cdot 0) \\ (1 \cdot 0 + 0 \cdot 1) & (1 \cdot 1 + 0 \cdot 0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

However, the conclusion fails: $T_P(T_A(x)) = b$ reads

$$T_P(T_A(x)) = T_P\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} x\right) = T_P\left(\begin{pmatrix} 2x \\ x \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2x \\ x \end{pmatrix} = \begin{pmatrix} 0 \cdot 2x + 1 \cdot x \\ 1 \cdot 2x + 0 \cdot x \end{pmatrix} = \begin{pmatrix} x \\ 2x \end{pmatrix}$$

which gives $x = 2, 2x = 4 \neq 1$, thus the system $Bx = b$ has no solution at all and our counter-example is finished - claim 1 is false.

Part 2:

Part 2 kindly asks us to study the linear transformation $T_A \circ T_P$, since $B = AP$ gives $T_B = T_A \circ T_P$.

Do we know that there exists exactly one $x \in \mathbb{R}^n$ s.t. $T_A(T_P(x)) = b$?

This time, we do.

We pick up our bow and shoot the first arrow: we know that there is exactly one $v \in \mathbb{R}^n$ with $T_A(v) = b$.

We shoot the second arrow: since P is invertible, $T_P : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is bijective.

Now let $x \in \mathbb{R}^n$.

$$Bx = b \iff T_A(T_P(x)) = b \underset{\text{from the uniqueness of } v}{\iff} T_P(x) = v$$

But since T_P is bijective and $v \in \mathbb{R}^n$, there exists a unique $x \in \mathbb{R}^n$ s.t. $T_P(x) = v$, i.e. $x = T_P^{-1}(v)$. So this x is the one and only solution, and our dragon is slain - claim 2 is true.

This is an exam problem I translated and solved. The original exam was written by the Linear Algebra 1 course lecturers at HUJI.