Semester 2026A, Exam A, Q1

Problem. Let V, W be vector spaces, let $T\colon V \to W$ be a linear map, let $\mathcal{B}, \mathcal{C}$ be bases of V and W respectively, and let $v \in V$. Prove that

$$[T]_{\mathcal{C}}^{\mathcal{B}} [v]_{\mathcal{B}} = [T(v)]_{\mathcal{C}}$$

Proof. Let us mark $\mathcal{B} = (v_1, \ldots, v_n)$, $\mathcal{C} = (w_1, \ldots, w_m)$ and $v = a_1 v_1 + \cdots + a_n v_n$.

$$[T]_{\mathcal{C}}^{\mathcal{B}} [v]_{\mathcal{B}} = \begin{pmatrix} | & & | \\ [T(v_1)]_{\mathcal{C}} & \cdots & [T(v_n)]_{\mathcal{C}} \\ | & & | \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = a_1 [T(v_1)]_{\mathcal{C}} + \cdots + a_n [T(v_n)]_{\mathcal{C}}$$

Let us recall that $[C_1]_{\mathcal{D}} + [C_2]_{\mathcal{D}} = [C_1 + C_2]_{\mathcal{D}}$ and that $a_1 [C_1]_{\mathcal{D}} = [a_1 C_1]_{\mathcal{D}}$. Thus

$$= [a_1 T(v_1) + \ldots a_n T(v_n)]_{\mathcal{C}} \underset{\text{T is linear}}{=} [T(a_1 v_1 + \cdots + a_n v_n)]_{\mathcal{C}} = [T(v)]_{\mathcal{C}}$$

□

This is an exam problem I translated and solved. The original exam was written by the Linear Algebra 1 course lecturers at HUJI.