Semester 2026A, Exam A, Q9

Problem. Are the following matrices similar?

$$A = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \in M_{5\times 5}(\mathbb{R})$$

Proof. Let us recall the claim that if matrices are similar, they have the same rank; thus, if matrices have different ranks, they are not similar.

$$\text{rkA} = \text{rk } T_A = \dim\left(Ax \mid x \in \mathbb{R}^5\right)$$

$$\text{rkB} = \text{rk } T_B = \dim\left(Bx \mid x \in \mathbb{R}^5\right)$$

For $x \in \mathbb{R}^5$,

$$Ax = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 + x_3 - x_4 + x_5 \\ -x_1 + x_2 - x_3 + x_4 - x_5 \\ x_1 - x_2 + x_3 - x_4 + x_5 \\ -x_1 + x_2 - x_3 + x_4 - x_5 \\ x_1 - x_2 + x_3 - x_4 + x_5 \end{pmatrix}$$

so

$$\text{Im}\left(T_A\right) = \left\{ t \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \mid t = x_1 - x_2 + x_3 - x_4 + x_5 \in \mathbb{R} \right\}$$

And since the scalar $t = x_1 - x_2 + x_3 - x_4 + x_5 \in \mathbb{R}$ attains every value in $\mathbb{R}$,

$$= \text{Span}\left(\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right)$$

A sequence of one non zero vector is always linearly independent, hence this sequence is a base of $\text{Im}\left(T_A\right)$, so $\text{rk}A = 1$.

For $x \in \mathbb{R}^5$,

$$Bx = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 + x_3 + x_5 \\ x_2 + x_4 \\ x_1 + x_3 + x_5 \\ x_2 + x_4 \\ x_1 + x_3 + x_5 \end{pmatrix}$$

so

$$\mathrm{Im}\,(T_B) = \left\{ t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + k \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \mid t = x_1 + x_3 + x_5 \in \mathbb{R}, k = x_2 + x_4 \right\}$$

These scalars reach every value in $\mathbb{R}$ so

$$= \mathrm{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

It's obvious this sequence is linearly independent (recall that a sequence of two vectors is linearly dependent if and only if one vector is a scalar multiple of the other):

$$\forall x \in \mathbb{R} \quad x \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ x \\ 0 \\ x \end{pmatrix} \neq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \wedge \quad \forall x \in \mathbb{R} \quad x \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ x \\ 0 \\ x \\ 0 \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

Therefore it's a basis for $\mathrm{Im}\,(T_B)$, and $\mathrm{rk}B = 2 \neq \mathrm{rk}A$ which means these matrices are not similar. □

This is an exam problem I translated and solved. The original exam was written by the Linear Algebra 1 course lecturers at HUJI.