Semester 2025B, Exam B, Q1

Problem. Let V be a finitely generated vector space such that dim $V = n$, and let $(v_1, \ldots, v_n)$ be a sequence of vectors from V. Prove that $(v_1, \ldots, v_n)$ is linearly independent if and only if $(v_1, \ldots, v_n)$ spans V.

Proof. We prove the two implications separately.

($\Rightarrow$):
Let us assume that $(v_1, \ldots, v_n)$ is linearly independent. That means that the only solution to the equation $a_1v_1 + \cdots + a_nv_n = 0$ s.t $a_1, \ldots, a_n \in \mathbb{F}$ is $a_1 = \cdots = a_n = 0$.
Let us assume for the sake of contradiction that $(v_1, \ldots, v_n)$ doesn't span V.
In that case, $\exists w \in V \ s.t \ w \notin$ Span $(v_1, \ldots, v_n)$.
In other words, w is not a linear combination of this sequence.
Recall the following: if a vector is not a linear combination of a sequence, and if the sequence is linearly independent, then we can add it to the sequence and the new sequence will also be linearly independent - which means $(v_1, \ldots, v_n, w)$ is still linearly independent.
On the other hand, recall that if dim $V = n$ and a sequence of vectors is of length $k > n$, then it must be linearly dependent.
We concluded that $(v_1, \ldots, v_n, w)$, a sequence of vectors from V of length $n + 1 > n$ is still linearly independent, which is a contradiction; so $(v_1, \ldots, v_n)$ must span V, as required.

($\Leftarrow$):
Let us assume that $(v_1, \ldots, v_n)$ spans V and assume for the sake of contradiction that $(v_1, \ldots, v_n)$ is linearly dependent.
Recall that if a sequence is linearly dependent, one of its vectors can be presented as a linear combination of the other vectors.
Then $\exists 1 \leq i \leq n$ s.t v_i is a linear combination of $\hat{v} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$.
Recall that if a sequence of vectors spans V and is linearly dependent, we can remove a vector that is a linear combination of the other vectors and the sequence will maintain its spanning property.
Therefore, $\hat{v}$ spans V.
However, $\hat{v}$ is of length $n - 1$, and another standard result specifies that for every sequence $\bar{v}$ which spans V, dim $V \leq len(\bar{v})$. This means $n \leq n - 1$ which is a contradiction; therefore $(v_1, \ldots, v_n)$ must be linearly independent, as required. □

My thanks to one ordered 100-tuple

This is an exam problem I translated and solved. The original exam was written by the Linear Algebra 1 course lecturers at HUJI.