Semester 2026A, Exam A, Q4

Problem. Let V be a vector space and let $v_1, \ldots, v_n, u, w \in V$ be such that the sequence $(v_1, v_2, \ldots, v_n, u)$ is linearly dependent, while the sequence $(v_1, \ldots, v_n, w)$ is linearly independent.
Is it necessarily the case that $w \notin \text{Span}(v_1, v_2, \ldots, v_n, u)$?

Proof. The sequence $(v_1, \ldots, v_n, w)$ is linearly independent, which means that the sequence $(v_1, \ldots, v_n)$ is also linearly independent (recall that if a sequence is linearly dependent one of the vectors is a linear combination of the others, hence a sequence that is already linearly dependent will stay linearly dependent when we add a vector to it).
Recall that if you add a vector to a linearly independent sequence which does not belong to the span of this sequence, you receive a new sequence which is still linearly independent, hence since $(v_1, v_2, \ldots, v_n, u)$ is linearly dependent we get that

$$u \in \text{Span}(v_1, v_2, \ldots, v_n)$$

Recall that if a vector belongs to the span of a sequence, the span of this sequence is equal to the span of this sequence along with said vector, hence

$$\text{Span}(v_1, v_2, \ldots, v_n) = \text{Span}(v_1, v_2, \ldots, v_n, u)$$

Which means w cannot belong to $\text{Span}(v_1, v_2, \ldots, v_n, u)$, because in that

$$w \in \text{Span}(v_1, v_2, \ldots, v_n)$$

which would mean the sequence $(v_1, \ldots, v_n, w)$ is linearly dependent. □

This is an exam problem I translated and solved. The original exam was written by the Linear Algebra 1 course lecturers at HUJI.