Semester 2026A, Exam B, Q3

Problem. Let V be a vector space over $\mathbb{R}$ and let (v_1, v_2, v_3) be a linearly independent sequence of vectors from V. Determine for which values $a \in \mathbb{R}$ the sequence

$$S = (v_1 + av_2,\ v_2 + av_3,\ v_3 + av_1)$$

is also linearly independent. Prove your answer.

Proof. Let us recall the theorem which states that a sequence of vectors is linearly independent if and only if the only way to reach 0 as a linear combination of it is by taking all of the scalars as zero.
Let us examine the following equation

$$\alpha_1 (v_1 + av_2) + \alpha_2 (v_2 + av_3) + \alpha_3 (v_3 + av_1) = 0$$
$$\alpha_1 v_1 + \alpha_1 a v_2 + \alpha_2 v_2 + \alpha_2 a v_3 + \alpha_3 v_3 + \alpha_3 a v_1 = 0$$
$$(\alpha_1 + \alpha_3 a)\, v_1 + (\alpha_1 a + \alpha_2)\, v_2 + (\alpha_2 a + \alpha_3)\, v_3 = 0$$

And from the same theorem, since (v_1, v_2, v_3) is linearly independent, we know that

$$\begin{cases} (\alpha_1 + \alpha_3 a) = 0 \\ (\alpha_1 a + \alpha_2) = 0 \\ (\alpha_2 a + \alpha_3) = 0 \end{cases}$$

We will use this linear system to determine which a values give us $\alpha_1 = \alpha_2 = \alpha_3 = 0$, i.e. that the only solution is the trivial solution.

$$\left(\begin{array}{ccc|c} 1 & 0 & a & 0 \\ a & 1 & 0 & 0 \\ 0 & a & 1 & 0 \end{array}\right) \overset{R_2 \to R_2 - aR_1}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & a & 0 \\ 0 & 1 & -a^2 & 0 \\ 0 & a & 1 & 0 \end{array}\right) \overset{R_3 \to R_3 - aR_2}{\Longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & a & 0 \\ 0 & 1 & -a^2 & 0 \\ 0 & 0 & 1 + a^3 & 0 \end{array}\right)$$

$\left(1 + a^3\right) \alpha_3 = 0$ means $\alpha_3 = 0$ or $a^3 = -1 \iff a = -1$.
Considering this, $\alpha_2 - a^2 \alpha_3 = 0$ means that if $\alpha_3 = 0$ then $\alpha_2 = 0$ and if $a = -1$ then $\alpha_2 = \alpha_3$.
Similarly, $\alpha_1 + a\alpha_3 = 0$ means that if $\alpha_3 = 0$ then $\alpha_1 = 0$ and if $a = -1$ then $\alpha_1 = \alpha_3$.
Which means that if $a \neq -1$ the set of solutions is $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$ and if $a = -1$ it's $\left\{ \begin{pmatrix} x \\ x \\ x \end{pmatrix} \mid x \in \mathbb{R} \right\}$,
which means that the sequence is linearly independent for $a \in \mathbb{R} \setminus \{-1\}$. □

This is an exam problem I translated and solved. The original exam was written by the Linear Algebra 1 course lecturers at HUJI.