

Semester 2026A, Exam A, Q3

Problem. For each $a \in \mathbb{R}$ we will define a system of linear equations over $\mathbb{R}$ by

$$\begin{cases} 7x + 2ay + z &= a \\ 8x + 3ay + z &= a \\ 10x + 5ay + 2z &= 2a \end{cases}$$

Determine, for each value of a , whether the system has no solution, a unique solution, or infinitely many solutions. In the case where at least one solution exists, write the solution set in parametric form.

We shall represent the system by its augmented matrix and reduce it to RREF using elementary row operations, keeping a as a parameter.

$$\begin{aligned} & \left[\begin{array}{ccc|c} 7 & 2a & 1 & a \\ 8 & 3a & 1 & a \\ 10 & 5a & 2 & 2a \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 10 & 5a & 2 & 2a \\ 8 & 3a & 1 & a \\ 7 & 2a & 1 & a \end{array} \right] \xrightarrow{R_1 \rightarrow \frac{1}{10}R_1} \left[\begin{array}{ccc|c} 1 & \frac{1}{2}a & \frac{1}{5} & \frac{1}{5}a \\ 8 & 3a & 1 & a \\ 7 & 2a & 1 & a \end{array} \right] \\ & \xrightarrow{\substack{R_2 \rightarrow R_2 - 8R_1 \\ R_3 \rightarrow R_3 - 7R_1}} \left[\begin{array}{ccc|c} 1 & \frac{1}{2}a & \frac{1}{5} & \frac{1}{5}a \\ 0 & (3 - \frac{8}{5})a & 1 - \frac{8}{5} & (1 - \frac{8}{5})a \\ 0 & (2 - \frac{7}{5})a & 1 - \frac{7}{5} & (1 - \frac{7}{5})a \end{array} \right] = \left[\begin{array}{ccc|c} 1 & \frac{1}{2}a & \frac{1}{5} & \frac{1}{5}a \\ 0 & -a & -\frac{3}{5} & -\frac{3}{5}a \\ 0 & -\frac{3}{2}a & -\frac{2}{5} & -\frac{2}{5}a \end{array} \right] \xrightarrow{\substack{R_2 \rightarrow -\frac{1}{a}R_2 \\ R_3 \rightarrow -\frac{1}{a}R_3 (*)}} \left[\begin{array}{ccc|c} 1 & \frac{1}{2}a & \frac{1}{5} & \frac{1}{5}a \\ 0 & 1 & \frac{3}{5a} & \frac{3}{5} \\ 0 & \frac{3}{2} & \frac{2}{5a} & \frac{2}{5} \end{array} \right] \end{aligned}$$

(*) this step assumes that $a \neq 0$, since we divide by a . The case $a = 0$ is handled separately below.

$$\begin{aligned} & \xrightarrow{\substack{R_1 \rightarrow R_1 - \frac{1}{2}aR_2 \\ R_3 \rightarrow R_3 - \frac{3}{2}R_2}} \left[\begin{array}{ccc|c} 1 & 0 & (\frac{1}{5} - \frac{a}{2} \cdot \frac{3}{5a}) & (\frac{1}{5}a - \frac{a}{2} \cdot \frac{3}{5}) \\ 0 & 1 & \frac{3}{5} & \frac{3}{5}a \\ 0 & 0 & (\frac{2}{5a} - \frac{3}{2} \cdot \frac{2}{5a}) & (\frac{2}{5} - \frac{3}{2} \cdot \frac{2}{5}) \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 0 & (\frac{1}{5} - \frac{3a}{10a}) & (\frac{1}{5}a - \frac{3a}{10}) \\ 0 & 1 & \frac{3}{5} & \frac{3}{5}a \\ 0 & 0 & (\frac{2}{5a} - \frac{9}{10a}) & (\frac{2}{5} - \frac{9}{10}) \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 0 & (-\frac{1}{10}) & (-\frac{1}{10}a) \\ 0 & 1 & \frac{3}{5} & \frac{3}{5}a \\ 0 & 0 & (-\frac{1}{2a}) & (-\frac{1}{10}) \end{array} \right] \\ & \xrightarrow{R_3 \rightarrow -2a \cdot R_3} \left[\begin{array}{ccc|c} 1 & 0 & (-\frac{1}{10}) & (-\frac{1}{10}a) \\ 0 & 1 & \frac{3}{5} & \frac{3}{5}a \\ 0 & 0 & 1 & a \end{array} \right] \xrightarrow{\substack{R_1 \rightarrow R_1 + \frac{1}{10}R_3 \\ R_2 \rightarrow R_2 - \frac{3}{5}R_3}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a \end{array} \right] \end{aligned}$$

Which gives the solution

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} \right\}$$

Therefore, for each $a \in \mathbb{R} \setminus \{0\}$, there is a unique solution.

Case $a = 0$. we return to the matrix obtained just before (*) and substitute $a = 0$:

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2}a & \frac{1}{5} & \frac{1}{5}a \\ 0 & -a & -\frac{3}{5} & -\frac{3}{5}a \\ 0 & -\frac{3}{2}a & -\frac{2}{5} & -\frac{2}{5}a \end{array} \right] \xrightarrow{a=0} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & -\frac{3}{5} & 0 \\ 0 & 0 & -\frac{2}{5} & 0 \end{array} \right] \xrightarrow{\substack{R_2 \rightarrow -5R_2 \\ R_3 \rightarrow -5R_3}} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right] \xrightarrow{\substack{R_1 \rightarrow R_1 - \frac{1}{15}R_2 \\ R_3 \rightarrow R_3 - \frac{2}{3}R_2 \text{ then } R_2 \rightarrow \frac{1}{3}R_2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

and the set of solutions is therefore

$$\left\{ \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} : y \in \mathbb{R} \right\}$$

so the system has infinitely many solutions.

To conclude,

$$\begin{cases} 0 \text{ solutions} & \text{No such } a \\ 1 \text{ solution} & a \in \mathbb{R} \setminus \{0\} \\ \infty \text{ solutions} & a \in \{0\} \end{cases}$$

This is an exam problem I translated and solved. The original exam was written by the Linear Algebra 1 course lecturers at HUJI.