

Result 1.45

Lemma. *condition for a direct sum*

Suppose $V_1, \dots, V_m$ are sub-spaces of V . Then $V_1 + \dots + V_m$ is a direct sum if and only if the only way to write 0_V as a sum $v_1 + \dots + v_m$ s.t. $\forall 1 \leq k \leq m, v_k \in V_k$ is with $\forall 1 \leq k \leq m, v_k = 0_V$.

Proof. We prove the two implications separately.

First direction:

Suppose $V_1 + \dots + V_m$ is a direct sum.

Then, in particular, $0 \in V_1 + \dots + V_m$ can be written in only one way as a sum $v_1 + \dots + v_m = 0$ where $\forall 1 \leq k \leq m, v_k \in V_k$.

Since $V_1, \dots, V_m$ are all sub-spaces of V , they all contain 0 (i.e. 0_V).

And since $0 + \dots + 0 = 0$, we get that this is the one way as needed.

Second direction:

Let us assume $0 + \dots + 0 = 0$ is the only way to write 0 as a sum $v_1 + \dots + v_m$ s.t. $\forall 1 \leq k \leq m, v_k \in V_k$. Then for every $a \in V_1 + \dots + V_m$, if $v_1 + \dots + v_m = a$ s.t. $\forall 1 \leq k \leq m, v_k \in V_k$ and $v'_1 + \dots + v'_m = a$ s.t. $\forall 1 \leq k \leq m, v'_k \in V_k$, we get that

$$v_1 + \dots + v_m - (v'_1 + \dots + v'_m) = a - a = 0$$

$$\therefore (v_1 - v'_1) + (v_2 - v'_2) + \dots + (v_m - v'_m) = 0$$

each V_k is a sub-space, hence closed under addition and scalar multiplication, therefore for each $1 \leq k \leq m$,

$(v_k - v'_k) = v_k + (-1) \cdot v'_k \in V_k$, so the assumption dictates that

$$v_1 - v'_1 = 0 \wedge v_2 - v'_2 = 0 \wedge \dots \wedge v_m - v'_m = 0$$

$$\therefore v_1 = v'_1 \wedge v_2 = v'_2 \wedge \dots \wedge v_m = v'_m$$

so there is exactly one way to write a in this manner ($a \in V_1 + \dots + V_m$ definitionally means one exists) and $V_1 + \dots + V_m$ is a direct sum as needed, hence the notation $V_1 \oplus \dots \oplus V_m$ can be used. $\square$