

Exercises 2B, Q1

Exercise. Determine all vector spaces that have exactly one basis, up to reordering, and prove that no others exist. Note that this is different from the original question.

Theorem. A vector space V over $\mathbb{F}$ has exactly one basis, up to reordering, if and only if it satisfies one of the two following statements:

1. $V = \{0\}$ over any field $\mathbb{F}$
2. $\mathbb{F}$ is isomorphic to $\mathbb{Z}_2$ and $\exists 0 \neq v$ s.t. $V = \{0, v\}$.

Remark. In the book, a basis is defined (1.8) as a list, which is a finite, ordered object.

Therefore, exactly one basis, up to reordering, means that every basis of the vector space is a permutation of a single list.

Remark. If you did not learn about isomorphism and prime fields, you can think of “ $\mathbb{F}$ is isomorphic to $\mathbb{Z}_2$ ” as $\mathbb{F} = \{0, 1\}$ with $1 + 1 = 0$.

Lemma 1. If V isn't finite-dimensional, then V has no basis. In particular, V does not have exactly one basis, up to reordering.

Proof. According to definition 2.9, a vector space is finite-dimensional iff some list of vectors in it spans the space. Therefore, if it isn't finite-dimensional (and by definition 2.13 it is then called infinite-dimensional), no list of vectors in it spans the space.

Since by definition 2.26 a basis must span the space, no list of vectors can be a basis of the space, hence it has no bases and in particular doesn't have exactly one. $\square$

Lemma 2. $V = \{0\}$ over any field $\mathbb{F}$ has exactly one basis.

Proof. $\{0\}$ is a vector space over any field $\mathbb{F}$ - after definition 1.22, $\{0\}$ is mentioned as the simplest vector space.

Definition 2.4 asserts $\text{Span } () = \{0\}$.

Definition 2.15 asserts that the empty list is linearly independent.

Therefore, by definition 2.26, $()$ is a basis for $\{0\}$.

Any non-empty list of vectors in $\{0\}$ must contain the zero vector, which makes it linearly dependent (according to example 2.18).

Thus, no non-empty list is a basis of $\{0\}$ (by definition 2.26), so $()$ is its only basis. $\square$

Lemma 3. *Let V be a finite-dimensional vector space over field $\mathbb{F}$, and let $\mathcal{B} = (v_1, \dots, v_n)$ be a basis of V such that $n = 1$.*

Then V has exactly one basis, up to reordering, if and only if $\mathbb{F}$ is isomorphic to $\mathbb{Z}_2$.

Proof. If $\mathbb{F}$ is isomorphic to $\mathbb{Z}_2$, then $\mathbb{F} = \{0, 1\}$ with $1 + 1 = 0$, and

$$V = \text{Span } \mathcal{B} = \{av_1 \mid a \in \mathbb{F}\} = \{0, v_1\}$$

and there is no other basis other than $\mathcal{B} = (v_1)$:

Since $\mathcal{B}$ is linearly independent, $v_1 \neq 0$ (by example 2.18).

The empty list is not a basis, since by definition 2.4, $\text{Span } () = \{0\} \neq V$.

Every vector in V is 0 or v_1 , so every list of vectors in V would contain some combination of these two elements.

By example 2.18, any list containing 0, or containing v_1 more than once, is linearly dependent and therefore not a basis.

Hence, overall, the only possible basis remains (v_1) .

If $\mathbb{F}$ is not isomorphic to $\mathbb{Z}_2$, then $\mathbb{F}$ has more than two elements, since every field with exactly two elements is isomorphic to $\mathbb{Z}_2$. Hence, there exists $c \in \mathbb{F}$ with $c \notin \{0, 1\}$.

Because $c \notin \{0, 1\}$ and v_1 cannot be 0 ($\mathcal{B}$ would be linearly dependent), $cv_1 \neq v_1$:

Otherwise,

$$0 = v_1 - cv_1 \iff 0 = (1 - c)v_1 \xrightarrow[\text{Exercises 1B Q 2}]{\iff} 1 - c = 0 \iff c = 1$$

which is a contradiction.

Hence, $\{cv_1\} \neq \{v_1\}$, and we will show that $\mathcal{B}' = (cv_1)$ is also a basis for V (we have shown that they are different up to reordering).

Because v_1 isn't the zero vector and $c \neq 0$, the contra-positive of exercises 1B question 2 gives $cv_1 \neq 0$, and $\mathcal{B}'$ contains a single non-zero vector, hence it is linearly independent.

Additionally,

$$V = \text{Span } \mathcal{B} = \{av_1 \mid a \in \mathbb{F}\} = \{a \cdot 1 \cdot v_1 \mid a \in \mathbb{F}\} \underset{0 \neq c \in \mathbb{F}}{=} \left\{ a \cdot \frac{1}{c} \cdot c \cdot v_1 \mid a \in \mathbb{F} \right\} = \left\{ \frac{a}{c} \cdot cv_1 \mid a \in \mathbb{F} \right\} \subseteq \text{Span } \mathcal{B}'$$

and since $\text{Span } \mathcal{B}'$ is a subspace of V (by 2.6), the two sided inclusion means $\mathcal{B}'$ spans V , and therefore it is a basis for V .

This means that if $\mathbb{F}$ is not isomorphic to $\mathbb{Z}_2$, V has more than one basis, up to reordering. □

Lemma 4. Let V be a finite-dimensional vector space over field $\mathbb{F}$, and let $\mathcal{B} = (v_1, \dots, v_n)$ be a basis of V such that $n \geq 2$. Then $\mathcal{B}$ is not the only basis of V , up to reordering.

Proof. Let us look at $\mathcal{B}' = (v_1 + v_2, v_2, \dots, v_n)$ and let $v \in V$. According to definition 2.26, since $\mathcal{B}$ is a basis of V , $\mathcal{B}$ spans V . Therefore,

$$\exists a_1, a_2, a_3, \dots, a_n \in \mathbb{F} \text{ s.t. } v = a_1 v_1 + a_2 v_2 + a_3 v_3 + \dots + a_n v_n$$

We can select

$$\begin{cases} c_1 = a_1 & \in \mathbb{F} \\ c_2 = a_2 - a_1 & \in \mathbb{F} \\ c_3 = a_3 & \in \mathbb{F} \\ \vdots & \\ c_n = a_n & \in \mathbb{F} \end{cases}$$

we get

$$\begin{aligned} v &= a_1 v_1 + a_2 v_2 + a_3 v_3 + \dots + a_n v_n = a_1 v_1 + a_1 v_2 + a_2 v_2 - a_1 v_2 + a_3 v_3 + \dots + a_n v_n \\ &= c_1 (v_1 + v_2) + c_2 v_2 + c_3 v_3 + \dots + c_n v_n \in \text{Span } \mathcal{B}' \end{aligned}$$

Hence $\mathcal{B}'$ also spans V .

Let $b_1, b_2, b_3, \dots, b_n \in \mathbb{F}$ s.t.

$$b_1 (v_1 + v_2) + b_2 v_2 + b_3 v_3 + \dots + b_n v_n = 0$$

Then

$$b_1 v_1 + (b_1 + b_2) v_2 + \dots + b_n v_n = 0$$

According to definition 2.26, since $\mathcal{B}$ is a basis of V , $\mathcal{B}$ is linearly independent and

$$b_1 = 0, \quad (b_1 + b_2) = 0, \quad \dots, \quad b_n = 0$$

hence

$$b_1 = b_2 = \dots = b_n = 0$$

and according to definition 2.15, $\mathcal{B}'$ is linearly independent as well.

According to definition 2.26, $\mathcal{B}'$ is a basis of V .

$\mathcal{B}' \neq \mathcal{B}$, nor is it a reordering of $\mathcal{B}$: let us assume for the sake of contradiction that $v_1 + v_2 \in \mathcal{B}$.

Then $\exists k \in \{1, \dots, n\}$ s.t. $v_k = v_1 + v_2$.

If $k = 1$ or $k = 2$ then accordingly $v_1 + v_2 = v_1$ or $v_1 + v_2 = v_2$ hence v_2 or v_1 is the zero vector, contradicting the linear independence of $\mathcal{B}$ (according to example 2.18, any list containing the zero vector is linearly dependent).

If $k \geq 3$ then $\mathcal{B}$ is of the form $(v_1, v_2, \dots, v_1 + v_2, \dots, v_n)$ and that

$$1 \cdot v_1 + 1 \cdot v_2 + 0 \cdot v_3 + \dots + 0 \cdot v_{k-1} + (-1) \cdot v_k + 0 \cdot v_{k+1} + \dots + 0 \cdot v_n = 0$$

contradicting the linear independence of $\mathcal{B}$.

Hence $\mathcal{B}'$ contains a vector not in $\mathcal{B}$, so $\mathcal{B}'$ is a basis of V different from $\mathcal{B}$, up to reordering. □

Proof. **Tying it all together,**

Let V be a vector space over field $\mathbb{F}$.

$\Leftarrow$: Let us assume V satisfies one of the two following statements:

1. $V = \{0\}$ over any field $\mathbb{F}$
2. $\mathbb{F}$ is isomorphic to $\mathbb{Z}_2$ and $\exists 0 \neq v$ s.t. $V = \{0, v\}$.

In the first case, we proved in lemma 2 that it has exactly one basis, up to reordering.

In the second case, (v) is a basis of V and we proved in lemma 3 that V has exactly one basis, up to reordering.

$\Rightarrow$ via contra-positive : Let us assume that V satisfies neither of these statements, and cover all possible cases of V .

If V is not finite-dimensional, then we proved in lemma 1 that it doesn't have exactly one basis.

Otherwise, V is finite-dimensional, therefore it has a basis (2.31).

Let $\mathcal{B} = (v_1, \dots, v_n)$ be a basis of V .

If $n = 0$, then $V = \text{Span } \mathcal{B} = \text{Span } () = \{0\}$ and it contradicts the fact that V doesn't satisfy statement 1.

If $n = 1$ and $\mathbb{F}$ isn't isomorphic to $\mathbb{Z}_2$, then we proved in lemma 3 that V doesn't have exactly one basis.

If $n = 1$ and $\mathbb{F}$ is isomorphic to $\mathbb{Z}_2$, then

$$V = \text{Span } (v_1) = \{av_1 \mid a \in \mathbb{F}\} = \{0 \cdot v_1, 1 \cdot v_1\} = \{0, v_1\}$$

and $v_1 \neq 0$ because $\mathcal{B}$ is a basis and is therefore linearly independent - in contradiction to the fact that V doesn't satisfy statement 2.

If $n \geq 2$, then we proved in lemma 4 that V doesn't have exactly one basis.

This covers all of the possibilities, as required. □