

Fermat's Theorem

Theorem. Let $f : (a, b) \rightarrow \mathbb{R}$ be a function from (a, b) to $\mathbb{R}$ and suppose that $x_0 \in (a, b)$ is a point where f has a local extremum. If f is differentiable at x_0 , then $f'(x_0) = 0$.

Proof. Since x_0 is a local extremum point of f , it's either a local maximum point or a local minimum point. Let us assume x_0 is a local maximum point. Then since it's also in particular an interior point,

$$\exists \delta > 0 \quad \text{s.t.} \quad \forall x \in (x_0 - \delta, x_0 + \delta) \subseteq (a, b), f(x) \leq f(x_0)$$

Since f is differentiable at x_0 it's in particular differentiable from the right and from the left while maintaining

$$f'_-(x_0) = f'(x_0) = f'_+(x_0)$$

We shall examine each one-sided derivative of f at x_0 in turn.

For every $x \in (x_0 - \delta, x_0)$ we get $f(x) \leq f(x_0)$ and $x < x_0$, which gives

$$\forall x \in (x_0 - \delta, x_0), \frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

Let us recall that limits weakly preserve inequalities. Hence

$$f'_-(x_0) = \lim_{x \rightarrow x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

For every $x \in (x_0, x_0 + \delta)$ we get $f(x) \leq f(x_0)$ and $x > x_0$, which gives

$$\forall x \in (x_0, x_0 + \delta), \frac{f(x) - f(x_0)}{x - x_0} \leq 0$$

And again since limits weakly preserve inequalities,

$$f'_+(x_0) = \lim_{x \rightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \leq 0$$

This gives us

$$0 \leq f'(x_0) \leq 0$$

Hence

$$f'(x_0) = 0$$

As required.

Now let us assume x_0 is a local minimum of f .

Let us recall that c is a local minimum of f if and only if c is a local maximum of $-f$.

Let us also recall that by the differentiable functions arithmetic theorem, f is differentiable at point c and $f'(c) = \alpha$ if and only if $-f$ is differentiable at point c and $(-f)'(c) = -\alpha$.
Let us notice that $(-f)$ fulfills the requirements for the case we just proved, hence the derivative of $-f$ at point x_0 is zero, which in turn means $f'(x_0) = 0$, as required. $\square$