

The intermediate value theorem

Theorem. Let $a, b, \gamma \in \mathbb{R}$ s.t. $a < b$ and let $f : D \rightarrow \mathbb{R}$ be a continuous function s.t. $[a, b] \subseteq D$ and $f(a) \neq f(b)$. If $f(a) < \gamma < f(b)$ or $f(b) < \gamma < f(a)$ then there exists $c \in (a, b)$ s.t. $f(c) = \gamma$.

Proof. Let us define $g : D \rightarrow \mathbb{R}$ by $g(x) = f(x) - \gamma$.

We will first prove that $\exists x_0 \in (a, b)$ s.t. $g(x_0) = 0$.

g is continuous as a shift of a continuous function.

Let us assume that $f(a) < \gamma < f(b)$. Then

$$g(a) = f(a) - \gamma < 0 < f(b) - \gamma = g(b)$$

Let us define the two sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ inductively.

$$a_1 = a, b_1 = b.$$

For $n \in \mathbb{N}$,

We will assume that $g\left(\frac{a_n + b_n}{2}\right) \neq 0$, otherwise $x_0 = \frac{a_n + b_n}{2}$ and the theorem holds.

If $g\left(\frac{a_n + b_n}{2}\right) < 0$ we will define $a_{n+1} = \frac{a_n + b_n}{2}$ and $b_{n+1} = b_n$, otherwise $g\left(\frac{a_n + b_n}{2}\right) > 0$ and we will define $a_{n+1} = a_n$ and $b_{n+1} = \frac{a_n + b_n}{2}$.

By induction, the following properties are true for each $n \in \mathbb{N}$:

1. $a_n \leq a_{n+1}$
2. $b_n \geq b_{n+1}$
3. $g(a_n) < 0 < g(b_n)$
4. $a \leq a_n < b_n \leq b$ hence $b_n - a_n > 0$.

We will now look at the sequence $(d_n)_{n=1}^{\infty}$ where $d_n = b_n - a_n$.

Let us notice that in the first case

$$b_{n+1} - a_{n+1} = b_n - \frac{a_n + b_n}{2} = \frac{2b_n - a_n - b_n}{2} = \frac{b_n - a_n}{2}$$

and in the second case

$$b_{n+1} - a_{n+1} = \frac{a_n + b_n}{2} - a_n = \frac{a_n + b_n - 2a_n}{2} = \frac{b_n - a_n}{2}$$

So

$$\forall n \in \mathbb{N}, d_{n+1} = \frac{d_n}{2}$$

and $(d_n)_{n=1}^{\infty}$ is a geometric sequence with $p = \frac{1}{2}$ and $d_1 = b - a$, hence

$$\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} \frac{b - a}{2^{n-1}} = 0$$

Points 1 and 2 and 4 give

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$$

and along with what we just proved, Cantor's intersection theorem provides a $x_0 \in [a, b]$ such that

$$x_0 = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$$

Therefore, according to Heine's sequence characterisation of continuity,

$$g(x_0) = \lim_{n \rightarrow \infty} g(a_n) \leq 0 \wedge g(x_0) = \lim_{n \rightarrow \infty} g(b_n) \geq 0$$

therefore

$$g(x_0) = 0$$

and because $g(a) < 0 < g(b)$, $x_0 \in (a, b)$.

According to the definition of g ,

$$f(x_0) - \gamma = 0 \Rightarrow f(x_0) = \gamma$$

as required.

If $f(b) < \gamma < f(a)$, we can apply the same proof for $-f$ and $-\gamma$ because $-f$ is also continuous and

$$-f(a) < -\gamma < -f(b)$$

□