

Weierstrass Extreme Value Theorem

Theorem. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function from $[a, b]$ to $\mathbb{R}$.
Then f attains a global maximum and a global minimum on $[a, b]$.

$$\exists c, d \in [a, b] : \forall x \in [a, b] \quad f(c) \leq f(x) \leq f(d)$$

Proof. We will first prove that f attains a global maximum.

According to the Weierstrass Boundedness Theorem, f is bounded on $[a, b]$.

Hence $\text{Im}(f)$ is a bounded subset of $\mathbb{R}$ and it is also a non-empty set (since $f(a) \in \text{Im}(f)$).

The upper bound theorem states that every non-empty subset of $\mathbb{R}$ which is bounded from above has a supremum, so $\text{Im}(f)$ has a supremum.

Denote $\beta = \sup(\text{Im}(f))$.

Also recall that since β is a supremum, it fulfills the supremum characterization

$$\forall \varepsilon > 0 \quad \exists y \in \text{Im}(f) : \beta - \varepsilon < y \leq \beta$$

And in particular,

$$\forall n \in \mathbb{N} \quad \exists y_n \in \text{Im}(f) : \beta - \frac{1}{n} < y_n \leq \beta$$

$$\therefore \forall n \in \mathbb{N} \quad \exists x_n \in [a, b] : \beta - \frac{1}{n} < f(x_n) \leq \beta$$

We shall examine $(x_n)_{n=1}^{\infty}$.

It is a bounded sequence, therefore according to the Bolzano-Weierstrass theorem, it has a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$.

Denote $\lim_{k \rightarrow \infty} x_{n_k} = d$.

Since limits of sequences weakly preserve inequalities maintained by all elements of the sequence, $d \in [a, b]$.

Because f is continuous, Heine's characterization of continuous functions dictates $\lim_{k \rightarrow \infty} f(x_{n_k}) = f(d)$.

Moreover, let us notice that since $n_k \geq k$ we get

$$\forall k \in \mathbb{N}, \quad \beta \geq f(x_{n_k}) > \beta - \frac{1}{n_k} \geq \beta - \frac{1}{k}$$

Since $\lim_{k \rightarrow \infty} (\beta - \frac{1}{k}) = \beta$, the squeeze theorem implies

$$f(d) = \lim_{k \rightarrow \infty} f(x_{n_k}) = \beta$$

Recall that a supremum which belongs to the set is always the maximum of the set, hence $\beta = \max(\text{Im}(f))$ and d is a global maximum point in $[a, b]$, as needed.

As for the global minimum: we know that a scalar multiple of a continuous function is also continuous, hence $-f$ is a continuous function, and the global maximum result applies to it:

$$\exists c \in [a, b] : \forall x \in [a, b] \quad -f(x) \leq -f(c)$$

$$\therefore \exists c \in [a, b] : \forall x \in [a, b] \quad f(x) \geq f(c)$$

This gives us the global minimum result. □