

Semester 2026A, Quiz 4

Abstract

Quiz 4 in the course Infinitesimal Calculus (1), semester 2026 A.

Quiz instructions: The duration of the quiz is 36 minutes.

The use of calculators and of any other auxiliary material is forbidden.

This is a quiz that I translated and solved.

The original quiz was written by the Infi Calculus 1 course lecturers at HUJI.

Problem (Question 1 - 65 points). Prove or disprove:

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function at $x_0 \in \mathbb{R}$ and in every punctured neighborhood of x_0 there exists a point $z \in \mathbb{R}$ s.t. $f(z) = z^2$, then $f(x_0) = x_0^2$.

Proof. using the $\varepsilon - \delta$ definition:

1. Since f is continuous at x_0 ,

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } \forall x \in (x_0 - \delta, x_0 + \delta) : |f(x) - f(x_0)| < \varepsilon$$

2. We shall define $g : \mathbb{R} \rightarrow \mathbb{R}$ by $g(x) = x^2$.

The function g is continuous on $\mathbb{R}$ because it is a polynomial, hence the same continuity characterisation as in point 1 applies to it.

3. Let us assume for the sake of contradiction that $f(x_0) \neq x_0^2$. Immediately, $f(x_0) \neq g(x_0)$.

4. The law of trichotomy establishes that either $f(x_0) > g(x_0)$ or $f(x_0) < g(x_0)$.

We will assume without loss of generality that $f(x_0) > g(x_0)$.

Hence, $f(x_0) - g(x_0) > 0$.

5. The previous point allows us to apply the known technique of halving the distance to separate these values and set $\varepsilon := \frac{f(x_0) - g(x_0)}{2} > 0$ in the property of points 1 and 2, which gives us corresponding deltas $\delta_1, \delta_2 > 0$.

We shall define $\delta_3 := \min\{\delta_1, \delta_2\}$ and get the combined property:

$$\forall x \in (x_0 - \delta_3, x_0 + \delta_3) \setminus \{x_0\} : |f(x) - f(x_0)| < \frac{f(x_0) - g(x_0)}{2} \wedge |g(x) - g(x_0)| < \frac{f(x_0) - g(x_0)}{2}$$

Expanding the absolute values gives:

$$\frac{g(x_0) - f(x_0)}{2} < f(x) - f(x_0) < \frac{f(x_0) - g(x_0)}{2} \wedge \frac{g(x_0) - f(x_0)}{2} < g(x) - g(x_0) < \frac{f(x_0) - g(x_0)}{2}$$

Isolating $f(x)$ and $g(x)$ on the relevant sides provides a lower bound for $f(x)$ and an upper bound for $g(x)$:

$$\frac{g(x_0) + f(x_0)}{2} < f(x) \wedge g(x) < \frac{f(x_0) + g(x_0)}{2}$$

Thus, in particular,

$$\forall x \in (x_0 - \delta_3, x_0 + \delta_3) \setminus \{x_0\} : g(x) < f(x)$$

6. This means that

$$\forall x \in (x_0 - \delta_3, x_0 + \delta_3) \setminus \{x_0\} : f(x) \neq x^2$$

In other words, $(x_0 - \delta_3, x_0 + \delta_3) \setminus \{x_0\}$ is a punctured neighborhood of x_0 which contradicts the assumption that every punctured neighborhood of x_0 contains a point $z \in \mathbb{R}$ s.t. $f(z) = z^2$. $\square$

Proof. using Heine's sequences:

1. By the assumption, for each $n \in \mathbb{N}$, there exists a point

$$x_n \in \left(x_0 - \frac{1}{n}, x_0 + \frac{1}{n}\right) \setminus \{x_0\} \quad \text{s.t.} \quad f(x_n) = x_n^2$$

Thus, we have a sequence $(x_n)_{n=1}^\infty$ such that for all $n \in \mathbb{N}$,

$$x_0 - \frac{1}{n} < x_n < x_0 + \frac{1}{n} \wedge x_n \neq x_0 \wedge f(x_n) = x_n^2$$

2. Since $\lim_{n \rightarrow \infty} (x_0 - \frac{1}{n}) = \lim_{n \rightarrow \infty} (x_0 + \frac{1}{n}) = x_0$, the squeeze theorem implies $\lim_{n \rightarrow \infty} x_n = x_0$.

3. Points 1 and 2 show that $(x_n)_{n=1}^\infty$ satisfies the conditions of a Heine sequence for continuity. By the assumption, f is continuous at x_0 , therefore $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$.

4. According to the arithmetic of sequence limits,

$$\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} x_n^2 = \lim_{n \rightarrow \infty} x_n \cdot \lim_{n \rightarrow \infty} x_n = x_0 \cdot x_0 = x_0^2$$

5. Therefore, $f(x_0) = x_0^2$, as required. □

Problem (Question 2 - 35 points). Let $(a_n)_{n=1}^\infty$ be a sequence of natural numbers that satisfies $\lim_{n \rightarrow \infty} \frac{a_n}{n} = \infty$.

Prove that $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{a_n} = \infty$.

Proof. Since $\lim_{n \rightarrow \infty} \frac{a_n}{n} = \infty$, the arithmetic of sequence limits ensures $\lim_{n \rightarrow \infty} \left(1 + \frac{a_n}{n}\right) = \infty$.

Additionally, because $a_n \in \mathbb{N}$ and $\frac{1}{n} > -1$ for all $n \in \mathbb{N}$, Bernoulli's inequality holds and

$$\forall n \in \mathbb{N} : \left(1 + \frac{1}{n}\right)^{a_n} \geq 1 + \frac{a_n}{n}$$

Recall the slice theorem: for two sequences $(b_n)_{n=1}^\infty$ and $(c_n)_{n=1}^\infty$, if there exists $N \in \mathbb{N}$ such that for every $n > N$ the inequality $b_n \geq c_n$ holds, and $\lim_{n \rightarrow \infty} c_n = \infty$, then $\lim_{n \rightarrow \infty} b_n = \infty$.

Therefore, $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{a_n} = \infty$ as required. □