

A Problem on the Convergence of a Recursive Sequence Depending on Its Initial Value

Problem. Let $(a_n)_{n=1}^{\infty}$ be a sequence s.t. $\forall n \in \mathbb{N} \quad a_{n+1} = a_n \cdot (1 - a_n)$.
Prove that $(a_n)_{n=1}^{\infty}$ converges if and only if $a_1 \in [0, 1]$.

Proof. We will prove both directions separately.

($\Rightarrow$) Let us assume $(a_n)_{n=1}^{\infty}$ is a convergent sequence.

1. Let $\lim_{n \rightarrow \infty} a_n = L$.

Since the sequence converges, the shifted sequence $(a_{n+1})_{n=1}^{\infty}$ must also converge to L .

Taking the limit on both sides of the recurrence relation and applying sequence limits arithmetic gives

$$L = L \cdot (1 - L) \iff L = L - L^2 \iff L^2 = 0 \iff L = 0$$

2. Note the following:

$$a_{n+1} \leq a_n \iff a_n \cdot (1 - a_n) \leq a_n \iff a_n - a_n^2 \leq a_n \iff -a_n^2 \leq 0 \iff a_n^2 \geq 0$$

which is true for every $n \in \mathbb{N}$ - therefore $(a_n)_{n=1}^{\infty}$ is a monotonically decreasing sequence.

3. Let us recall that every convergent sequence is bounded, and that a monotonically decreasing sequence which is bounded from below converges to the infimum of its elements.

Therefore, 0 is the infimum of the sequence elements and in particular $a_1 \geq 0$.

4. If $a_1 = 0$, then $a_1 \leq 1$. Otherwise, $a_1 > 0$ and the infimum provides $a_2 \geq 0$.

Substituting the recurrence relation gives $a_1 - a_1^2 \geq 0 \iff a_1 \geq a_1^2 \iff 1 \geq a_1$.

5. Overall, $a_1 \in [0, 1]$ as required.

($\Leftarrow$) Assume $a_1 \in [0, 1]$.

By step 2 of the previous direction (this argument did not use convergence), $(a_n)_{n=1}^{\infty}$ is a monotonically decreasing sequence.

Let us prove by induction that $\forall n \in \mathbb{N} \quad a_n \in [0, 1]$.

For $n = 1$, the assumption provides $a_1 \in [0, 1]$.

Assume for some $n \in \mathbb{N}$ that $a_n \in [0, 1]$. Then,

$$0 \leq a_n \leq 1 \iff 0 \leq a_n^2 \leq a_n \iff -a_n \leq a_n^2 - a_n \leq 0 \iff a_n \geq a_n - a_n^2 \geq 0 \iff a_n \geq a_{n+1} \geq 0$$

The induction assumption provides $1 \geq a_n$ therefore $a_{n+1} \in [0, 1]$ and the induction is complete.

Let us recall that a sequence which is monotonically decreasing and is bounded from below is convergent, hence $(a_n)_{n=1}^{\infty}$ is a convergent sequence. $\square$