

The Bolzano-Weierstrass Theorem

Remark. When I write monotonically increasing or decreasing, I refer to the weak sense. For the strong sense, I add the word “strictly”.

Lemma. Every sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}$ has a monotone subsequence (a subsequence that is either non-decreasing or non-increasing).

Proof. Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$.
Let us look at

$$A = \{m \in \mathbb{N} : \forall n > m, a_m \geq a_n\}$$

This is commonly called the set of peaks of (a_n) .

Case 1: $A = \emptyset$ or A has an upper bound.

If $A = \emptyset$, we will take $j = 1$. Otherwise, we will assume $A \neq \emptyset$ and that A is bounded from above. Let $M \in \mathbb{R}$ be an upper bound of A , thus, the Archimedean property allows us to take a $j \in \mathbb{N}$ s.t. $j > M$.

Thus, every index that is bigger or equal to j is not in A , hence

$$\forall n \geq j, \exists n' > n, \text{ s.t. } a_n < a_{n'}$$

We shall inductively construct a subsequence. Our first index will be $n_1 = j$, and for every index $j \leq n_k$ the formula above provides a $n'_k > n_k \geq j$ such that $a_{n_k} < a_{n'_k}$, so we shall define n_{k+1} as this n'_k .

Then $(n_k)_{k=1}^{\infty}$ is a strictly monotonically increasing sequence of natural numbers, which means it successfully defines a subsequence of (a_n) , and again from our formula $\forall k \in \mathbb{N}, a_{n_k} < a_{n_{k+1}}$ which means this subsequence is strictly monotonically increasing, thus monotonically increasing in particular, as required.

Case 2: Otherwise, A has no upper bound and isn't empty.

This means that for every index $m \in A$, there exists an index $w > m$ s.t. $w \in A$.

We shall inductively construct a subsequence.

As a set defined by indexes and by our assumption, $\emptyset \neq A \subseteq \mathbb{N}$, so by the well-ordering principle of $\mathbb{N}$ it has a minimum.

Our first index will be $n_1 = \min(A)$, and for every index n_k such that $n_k \in A$, we know there is another index $p > n_k$ s.t. $p \in A$.

We shall define $n_{k+1} = p$.

Then $(n_k)_{k=1}^{\infty}$ is a strictly monotonically increasing sequence of natural numbers, which means it successfully defines a subsequence of (a_n) , and by the definition of A ,

$$\forall k \in \mathbb{N}, a_{n_{k+1}} \leq a_{n_k}$$

thus $(a_{n_k})_{k=1}^{\infty}$ is monotonically decreasing, as needed. □

Theorem (The Bolzano-Weierstrass theorem). *Every bounded sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}$ has a convergent subsequence.*

Proof. Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence in $\mathbb{R}$.

By the lemma we proved, there exists a subsequence $(a_{n_k})_{k=1}^{\infty}$ that is monotone, and it is also bounded since boundedness is inherited by subsequences. Let us recall that a sequence that is both monotone and is bounded converges (the Monotone Convergence Theorem), therefore $(a_{n_k})_{k=1}^{\infty}$ converges and our proof is finished. $\square$