

Semester 2026A, Exam A, Q1

Problem. Prove that a bounded sequence is converging if and only if it has exactly one partial limit.

Proof. We prove the two implications separately.

First direction:

Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence and let us assume that it's converging to $L \in \mathbb{R}$.

Existence: by the Bolzano-Weierstrass theorem, (a_n) has a converging sub-sequence.

Uniqueness: Let $(a_{n_k})_{k=1}^{\infty}$ be a converging sub-sequence of (a_n) .

Then from the inheritance theorem (every sub-sequence of a convergent sequence converges to the same limit), it also converges to L , as needed.

Second direction:

Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence such that it has exactly one partial limit, $L \in \mathbb{R}$.

Let us assume for the sake of contradiction that $(a_n)_{n=1}^{\infty}$ doesn't converge.

In particular, it doesn't converge to L , so there exists $\varepsilon_1 > 0$ such that $|a_n - L| > \varepsilon_1$ is a property that holds for infinitely many n (otherwise it would converge to L).

Since it's a frequent property of (a_n) , there exists a sub-sequence $(a_{n_j})_{j=1}^{\infty}$ of which all elements fulfill this property.

In particular, $(a_{n_j})_{j=1}^{\infty}$ inherits the boundedness, therefore by the Bolzano-Weierstrass theorem it has a converging sub-sequence $(a_{n_{j_w}})_{w=1}^{\infty}$, i.e. there exists $L' \in \mathbb{R}$ such that $\lim_{w \rightarrow \infty} a_{n_{j_w}} = L'$.

However, every element of this sub-sequence fulfills the property of being far from L by at least ε_1 , and since limits weakly preserve sequence inequalities, $|L' - L| \geq \varepsilon_1 > 0$, therefore $L' \neq L$.

L' is also a partial limit of (a_n) , since the sub-sequence of a sub-sequence is also a sub-sequence, which contradicts our assumption. $\square$

This is an exam problem I translated and solved. The original exam was written by the Infinitesimal Calculus 1 course lecturers at HUJI.