

Semester 2026B, Quiz 1

Abstract

Quiz 1 in the course Infinitesimal Calculus (2), semester 2026 B.

This is a quiz that I translated and solved.

The original quiz was written by the Infi Calculus 2 course lecturers at HUJI.

Problem 1. Find a $q \in \mathbb{Q}$ s.t. $|e^{-1} - q| < 10^{-2}$.

Solution. Our goal is to find a rational approximation for the function e^x evaluated at $x = -1$, with an error strictly less than 10^{-2} .

The closest point to $x = -1$ which we can easily evaluate is $x = 0$ since $e^0 = 1$.

The Maclaurin polynomial (the Taylor polynomial at 0) of degree n of e^x is

$$P_{n,e^x,0}(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k = e^0 + e^0 \cdot x + \frac{e^0}{2!} \cdot x^2 + \cdots + \frac{e^0}{n!} \cdot x^n = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

Let $n \in \mathbb{N}$. To apply Taylor's Theorem on e^x in the interval $[-1, 0]$, e^x must have a continuous n -th derivative on $[-1, 0]$ and be differentiable $n + 1$ times on $(-1, 0)$.

e^x is differentiable n times on $[-1, 0]$ with its n -th derivative being e^x , which is continuous on $\mathbb{R}$ and in particular on $[-1, 0]$.

e^x is also differentiable $n + 1$ times on $(-1, 0)$.

In fact, e^x is a smooth function (it is infinitely differentiable) on $\mathbb{R}$.

Hence, the Taylor Theorem with the Lagrange Remainder applies and there exists a point $c \in (-1, 0)$ such that

$$R_{n,e^x,0}(-1) = e^{-1} - P_{n,e^x,0}(-1) = \frac{e^c}{(n+1)!} (-1)^{n+1}$$

This means that

$$|e^{-1} - P_{n,e^x,0}(-1)| \leq \left| \frac{e^c}{(n+1)!} (-1)^{n+1} \right| = \left| \frac{e^c}{(n+1)!} \right| \stackrel{(*)}{\leq} \frac{1}{(n+1)!}$$

(*) since e^x is a monotonically increasing function, $\forall x \in (-1, 0)$, $e^x \leq e^0 = 1$.

To guarantee our error is strictly less than 10^{-2} , we need $\frac{1}{(n+1)!} < \frac{1}{100}$, which implies $(n+1)! > 100$.

Since $4! = 24$ and $5! = 120$, the minimum degree required is $n = 4$.

Therefore, setting the degree $n = 4$ gives the rational approximation:

$$P_{4,e^x,0}(-1) = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} = \frac{3}{8} \in \mathbb{Q}$$

which satisfies the error bound:

$$|e^{-1} - P_{4,e^x,0}(-1)| \leq \frac{1}{5!} < \frac{1}{100} = 10^{-2}$$

as required.

Problem 2. Find all antiderivatives of $f(x) = \frac{x^2-4x}{x^2-3x+2}$ on $(1, 2)$.

Solution. Since the degree of the numerator is greater than or equal to the degree of the denominator, our first step is to perform polynomial division with a remainder.

We can achieve this by expressing the numerator in terms of the denominator.

$$f(x) = \frac{x^2 - 3x + 2 - x - 2}{x^2 - 3x + 2} = 1 + \frac{-x - 2}{x^2 - 3x + 2}$$

Next, we focus on finding the antiderivatives of the remainder, $g(x) = \frac{-x-2}{x^2-3x+2}$, on the interval $(1, 2)$.

As a best practice, we first check if the numerator is a scalar multiple of the denominator's derivative, which would allow for direct logarithmic integration.

$$(x^2 - 3x + 2)' = 2x - 3$$

Since this is not the case, we proceed with partial fraction decomposition.

We first split the denominator into coprime polynomials.

$$x^2 - 3x + 2 = (x - 2)(x - 1)$$

Hence

$$\begin{aligned} g(x) &= \frac{-x - 2}{(x - 2)(x - 1)} \\ \frac{-x - 2}{(x - 2)(x - 1)} &= \frac{A}{x - 2} + \frac{B}{x - 1} \\ -x - 2 &= (x - 1)A + (x - 2)B \\ (A + B)x - A - 2B &= -x - 2 \end{aligned}$$

And since two polynomials are equal if and only if their corresponding coefficients are equal,

$$\begin{cases} A + B = -1 \\ -A - 2B = -2 \end{cases} \Rightarrow \begin{cases} A = -4 \\ B = 3 \end{cases}$$

$$g(x) = \frac{-4}{x - 2} + \frac{3}{x - 1}$$

Now we can return to our original function, $f(x) = 1 + g(x)$, and find the antiderivatives term by term.

The antiderivative of 1 is $F_1(x) = x + C$.

The antiderivative of $\frac{-4}{x-2}$ is

$$F_2(x) = -4 \ln |x - 2| + C$$

and the antiderivative of $\frac{3}{x-1}$ is

$$F_3(x) = 3 \ln |x - 1| + C$$

Therefore, the general antiderivative of the function is

$$F(x) = x - 4 \ln |x - 2| + 3 \ln |x - 1| + C$$

For $x \in (1, 2)$, we have $x - 2 < 0$ and $x - 1 > 0$. Thus, the antiderivative simplifies to:

$$F(x) = x - 4 \ln (2 - x) + 3 \ln (x - 1) + C$$